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# ON THE CONCAVITY OF DELIVERY GAMES

HERBERT HAMERS <sup>\*</sup> <sup>†</sup>

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## Abstract

Delivery games, introduced by *Hamers, Borm, van de Leensel and Tijs (1994)*, are combinatorial optimization games that arise from delivery problems closely related to the Chinese postman problem (CPP). They showed that delivery games are not necessarily balanced. For delivery problems corresponding to the class of bridge-connected Euler graphs they showed that the related games are balanced.

This paper focuses on the concavity property for delivery games. A delivery game arising from a delivery model corresponding to a bridge-connected Euler graph needs not to be concave. The main result will be that for delivery problems corresponding to the class of bridge-connected cyclic graphs, which is a subclass of the class of bridge-connected Euler graphs, the related delivery games are concave.

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# 1 Introduction

The delivery situations, introduced in *Hamers, Borm, van de Leensel and Tijs (1994)*, are defined on a connected graph in which a cost function is defined on the edges. For these situations, which are closely related to the Chinese postman problem (cf. *Mei-Ko Kwan (1962)*), they defined a new class of combinatorial optimization games called delivery games. They showed that these games need not to be balanced. For delivery situations corresponding to bridge-connected Euler graphs they showed that the related delivery games are balanced.

This paper focuses on the concavity property of delivery games. A game satisfies the concavity property if the characteristic function of a delivery game is a sub-modular function. In general a delivery game is not concave, since concavity would imply that the game is balanced. But even for the class of balanced games corresponding to the delivery problems that arise from bridge-connected Euler graphs concavity is not necessarily satisfied. We will show that for delivery games corresponding to the class of bridge-connected cyclic graphs, which is a subclass of the class of bridge-connected Euler graphs, the related delivery games are concave.

There are several arguments to ask for concavity. Concave games are balanced, which means that one can find core-elements prescribing an allocation of the worth of the grand coalition among the players in such a way that no subgroup has an incentive to split off. Moreover, *Shapley (1971)* (cf. *Edmonds (1970)*) and *Ishiichi (1980)* showed that the extreme points of the core are the marginal vectors of the game if and only if the game is concave. Here, a marginal vector allocates to each player the marginal contribution this player constitutes according to a given way (a permutation) to form the grand coalition. With respect to one-point game theoretical solution concepts, concave games are nice since the Shapley value (*Shapley (1953)*), which is by definition the average of all marginal vectors, is in the barycentre of the core. Further, the  $\tau$ -value (*Tijs (1981)*), which is an efficient compromise between an utopia vector and a minimal right vector, can be easily calculated.

Concavity, or its counterpart convexity (super-modular), is a rare property for combinatorial games. Only two classes of sequencing games, considered in *Curiel, Pedersen and Tijs (1989)* and *Hamers, Borm and Tijs (1992)*, are contained in the class of convex games. Other

combinatorial games, e.g. minimum spanning tree games (cf. *Granot and Huberman (1981)*) and traveling sales man games (cf. *Potters, Curiel and Tijs (1992)*), do not posses the concavity property.

The paper is organized as follows. In section 2 the delivery game is formulated and an example is provided that a delivery game corresponding to delivery problem that arises from a bridge-connected Euler graph is not necessarily concave. Then the main theorem is formulated. Section 3 proves the rather technical theorems and lemmata needed for the concavity result.

## 2 Delivery problems and delivery games

This section describes the class of delivery models and the corresponding delivery games as introduced in *Hamers, Borm, van de Leensel and Tijs (1994)*. Before the formal description is provided we need some definitions. Let  $G = (V, E, i)$  be an undirected connected graph where  $V$  is the set of vertices,  $E$  the set of edges and  $i$  the incidence mapping that assigns to each edge of  $E$  an unordered pair of not necessarily different vertices of  $V$ . A *walk* in  $G = (V, E, i)$  is a finite sequence of edges and vertices of the form  $v_1, e_1, v_2, \dots, e_k, v_{k+1}$  with  $k \geq 0, v_1, \dots, v_{k+1} \in V, e_1, \dots, e_k \in E$  such that  $i(e_j) = \{v_j, v_{j+1}\}$  for all  $j \in \{1, \dots, k\}$ . Such a walk is a *closed walk* if  $v_1 = v_{k+1}$ . A closed walk in which all edges are distinct is called a *closed trail*. A *path* in  $G$  is a walk in which all vertices (except, possibly  $v_1 = v_{k+1}$ ) and edges are distinct. A closed path, i.e.  $v_1 = v_{k+1}$ , containing at least one edge is called a *circuit*.

Let  $G = (V, E, i)$  be a connected graph. We assume that each edge corresponds to one player. Formally, there is a one-one map  $g : E \rightarrow N$ , where  $N = \{1, \dots, |E|\}$  is the set of players. Further, we pick a vertex  $v_0 \in V$  which is called the post office of  $G$ . We now define an  $S$ -tour of a coalition  $S$  as a closed walk that starts in the post office  $v_0$  and visits each edge that corresponds to a player of  $S$  at least once. Formally, we have

**Definition 1** *Let  $G = (V, E, i, v_0)$  be a connected undirected graph in which  $v_0 \in V$  is the post office. Then an  $S$ -tour is a closed walk  $v_0, e_1, v_1, \dots, v_{k-1}, e_k, v_0$  in  $G$  such that  $S \subset \{g(e_j) \mid j \in \{1, \dots, k\}\}$ .*

The set of  $S$ -tours of a coalition is denoted by  $D(S)$ .

To the edges of the graph  $G$  we assign deliver costs  $d : E \rightarrow [0, \infty)$  and travel costs

$t : E \rightarrow [0, \infty)$ . A postman who has to deliver the mail to a coalition  $S$  has to pick an  $S$ -tour  $v_0, e_1, v_1, \dots, v_{k-1}, e_k, v_0 \in D(S)$ . Each time the postman visits an edge the travel costs of this edge is charged. In case the postman makes a delivery in a street the deliver costs of this street is also charged. Formally, the costs of an  $S$ -tour  $v_0, e_1, v_1, \dots, v_{k-1}, e_k, v_0$  are equal to

$$C_S(v_0, e_1, \dots, v_{k-1}, e_k, v_0) = \sum_{j=1}^k t(e_j) + \sum_{e \in g^{-1}(S)} d(e) \quad (1)$$

We assume that in any  $S$ -tour each street of  $S$  pays its own specific deliver costs and also once his specific travel costs. Hence, each player of  $S$  has individual fixed costs. The sum of these fixed costs of the members of  $S$  is equal to  $\sum_{e \in g^{-1}(S)} (t(e) + d(e))$  and are called the non-separable costs of an  $S$ -tour. Note that the non-separable costs are independent of the chosen  $S$ -tour. We will call the remaining costs of an  $S$ -tour the separable costs of an  $S$ -tour. Consequently, we can rewrite expression (1) as the sum of non-separable costs and separable costs, i.e.

$$C_S(v_0, e_1, \dots, v_{k-1}, e_k, v_0) = \left[ \sum_{e \in g^{-1}(S)} (t(e) + d(e)) \right] + \left[ \sum_{j=1}^k t(e_j) - \sum_{e \in g^{-1}(S)} t(e) \right]. \quad (2)$$

Since each coalition  $S$  wants to be delivered as cheap as possible, it will choose an  $S$ -tour in  $D(S)$  that minimizes (2). Since the non-separable costs are independent of the chosen  $S$ -tour coalition  $S$  will choose an  $S$ -tour in  $D(S)$  that minimizes

$$T_S(v_0, e_1, \dots, v_{k-1}, e_k, v_0) = \sum_{j=1}^k t(e_j) - \sum_{e \in g^{-1}(S)} t(e). \quad (3)$$

We call such an  $S$ -tour a *minimal  $S$ -tour*.

In the following a delivery problem is denoted by  $\Gamma = (N, (V, E, i, v_0), t, g)$ . Here,  $N$  is the set of players,  $(V, E, i, v_0)$  is a connected graph in which  $v_0$  is the post office,  $t : E \rightarrow [0, \infty)$  assigns the travel costs to the edges and  $g$  gives the one-one correspondence between edges and players. The class of delivery problems corresponding to the player set  $N$  is denoted by  $DP(N)$ . Note that the function  $d : E \rightarrow [0, \infty)$  which assigns the deliver costs to the edges is omitted in the description of  $\Gamma$  since it does not affect the choice of a minimal tour (cf. (3)).

Before delivery games are formally introduced we recall some well known facts concerning cooperative games. A *cooperative cost game* is a pair  $(N, c)$  where  $N$  is a finite set of players and  $c$  is a mapping  $c : 2^N \rightarrow \mathbf{R}$  with  $c(\emptyset) = 0$  and  $2^N$  is the collection of all subsets of  $N$ . A game  $(N, c)$  is called *concave* if for all coalitions  $S, T \in 2^N$  and all  $i \in N$  with  $S \subset T \subset N \setminus \{i\}$  it holds that

$$c(T \cup \{i\}) - c(T) \leq c(S \cup \{i\}) - c(S).$$

Hence, a game  $(N, c)$  is concave if and only if the map  $c : 2^N \rightarrow \mathbf{R}$  is sub-modular. A core element  $x = (x_i)_{i \in N} \in \mathbf{R}^N$  is such that no coalition has an incentive to split of, i.e.

$$\sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S) \text{ for all } S \in 2^N.$$

The core  $Core(c)$  consists of all core elements. A game is called balanced if its core is non-empty. The results of *Shapley (1971)* and *Ishiichi (1980)* give that concavity implies balancedness.

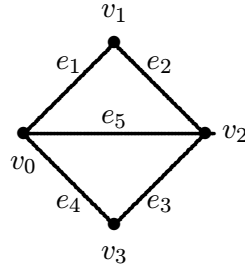
By defining the cost value of a coalition  $S$  as the separable costs of a minimal  $S$ -tour, we obtain a cooperative game corresponding to a delivery problem. This cooperative game is called the delivery game and is formal defined in following definition.

**Definition 2** *The delivery game  $(N, c)$  corresponding to  $\Gamma = (N, (V, E, i, v_0), t, g) \in DP(N)$  is defined for all  $S \subset N$  by*

$$c(S) := \min_{v_0, e_1, \dots, e_k, v_0 \in D(S)} \left( \sum_{j=1}^k t(e_j) - \sum_{e \in g^{-1}(S)} t(e) \right). \quad (4)$$

The following example illustrates a delivery game and shows that these games need not to be balanced.

**Example 1** Let  $V = \{v_0, v_1, v_2, v_3\}$ ,  $E = \{e_1, e_2, e_3, e_4, e_5\}$  and  $i$  is defined by  $i(e_1) = \{v_0, v_1\}$ ,  $i(e_2) = \{v_1, v_2\}$ ,  $i(e_3) = \{v_2, v_3\}$ ,  $i(e_4) = \{v_0, v_3\}$ ,  $i(e_5) = \{v_0, v_2\}$  (see figure



1).

figure 1

Let  $t(e) = 1$  for all  $e \in E$  and  $g(e_j) = j, j \in \{1, \dots, 5\}$ . Then  $c(N) = 1$  since  $v_0, e_1, v_1, e_2, v_2, e_3, v_3, e_4, v_0, e_5, v_2, e_5, v_0$  is a minimal  $N$ -tour with separable costs equal to  $t(e_5) = 1$ . For all  $S \in A := \{\{1, 2, 5\}, \{3, 4, 5\}, \{1, 2, 3, 4\}\}$  we have that  $c(S) = 0$ . Suppose  $x \in Core(c)$ , then

$$0 < 2 = 2c(N) = \sum_{S \in A} \sum_{i \in S} x_i \leq \sum_{S \in A} c(S) = 0.$$

Contradiction, so no core elements exists.

Hamers *et al.*(1994) showed that delivery games arising from delivery problems corresponding to bridge-connected Euler graphs are balanced. An edge  $b \in E$  is called a bridge of a connected graph  $G = (V, E, i)$  if the graph  $\hat{G} = (V, E - \{b\}, i|_{E - \{b\}})$  is a disconnected graph. A connected graph  $G = (V, E, i)$  is called a bridge-connected Euler graph if all the components of the graph  $\hat{G} = (V, E - B(G), i|_{E - B(G)})$  are Euler graphs. Here,  $B(G)$  is the set of bridges of  $G$ . The following example shows that delivery games arising from delivery games corresponding to bridge-connected Euler graphs need not to be concave.

**Example 2** Let  $K_7$  be the complete graph with seven vertices. Obviously, it is an Euler graph and, consequently a bridge-connected Euler graph. In figure 1 the notation  $e_1, 1$  means that edge  $e_1$  has travel costs equal to 1. The edges that are not drawn have travel costs equal to 100.

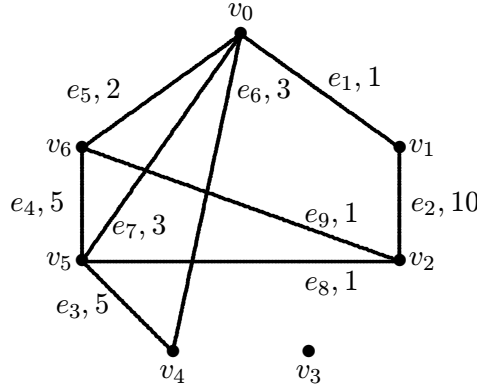


figure 2

Take the following coalitions:  $T = \{g(e_3), g(e_4), g(e_6)\}$ ,  $S = \{g(e_3)\}$  and  $\{i\} = \{g(e_2)\}$ . The minimal  $T$ -tour is equal to  $v_0, e_6, v_4, e_3, v_5, e_4, v_6, e_5, v_0$ . Hence,  $c(T) = t(e_5) = 2$ . The minimal  $T \cup \{i\}$ -tour is equal to  $v_0, e_6, v_4, e_3, v_5, e_4, v_6, e_9, v_2, e_2, v_1, e_1, v_0$ . Hence,  $c(T \cup \{i\}) = t(e_1) + t(e_9) = 2$ . The minimal  $S$ -tour is equal to  $v_0, e_6, v_4, e_3, v_5, e_7, v_0$ . Hence,  $c(S) = t(e_6) + t(e_7) = 6$ . The minimal  $S \cup \{i\}$ -tour is equal to  $v_0, e_6, v_4, e_3, v_5, e_8, v_2, e_2, v_1, e_1, v_0$ . Hence,  $c(S \cup \{i\}) = t(e_6) + t(e_8) + t(e_1) = 5$ . This implies that the delivery game is not concave, since  $c(T \cup \{i\}) - c(T) = 0 > -1 = c(S \cup \{i\}) - c(S)$ .  $\square$

The final part of this section states that delivery games corresponding to delivery problems arising from bridge-connected cyclic graphs are concave games. A graph  $G = (V, E, i)$  is called a bridge-connected cyclic graph if all components of the graph  $(V, E, i|_{E - B(G)})$  are union of circuits or single vertices. Hence, all circuits in such a graph are edge-disjoint. Note that

the class of bridge-connected cyclic graph is a subclass of the class of bridge-connected Euler graphs.

Before the formal proof of this concavity result is presented in the next section we will give a short description of the construction of this proof. First we show that if the underlying graph of the delivery problem is a circuit or the union of a circuit and a bridge then the corresponding delivery game is concave. Second we introduce bridge-connected circuit graphs. These are the graphs that after removing the bridges only have circuits or single vertices as components. Hence, here the circuits are vertex disjoint. It is shown with the help of the results on circuit (with a bridge) graphs and some reduction lemma's that also delivery games are concave that arise from delivery problems corresponding to bridge-connected circuit graphs. Finally, we proof concavity for bridge-connected cyclic graphs by giving an extension of a bridge-connected circuit graph to a bridge-connected cyclic graph.

**Theorem 1** *Each delivery game that arises from a delivery problem corresponding to a bridge-connected cyclic graph is a concave game.*

### 3 Proof main result

For the formal proof we need some new notations. Let  $DE(N)$  represent the delivery problems corresponding to a weakly cyclic graph. We assume in this section that  $\Gamma = (N, (V, E, i, v_0), t, g) \in DE(N)$ . Take  $j \in N$  and let  $e^* \in E$  be such that  $g(e^*) = j$ . Next, take  $S \subset T \subset N \setminus \{j\}$  and abbreviate a minimal delivery tour of coalition  $S, S \cup \{j\}$   $T$  and  $T \cup \{j\}$  by  $d_1, d_2, d_3$  and  $d_4$ , respectively.

Let  $\Gamma = (N, (V, E, i, v_0), t, g)$  and let  $U \subset N$ . Let  $d = v_0, e_1, \dots, e_k, v_0 \in D(U)$ , then the separable travel costs of the delivery tour  $d$  with respect to  $U$  and a subset of edges  $E' \subset E$  is given by a function  $f_d^{\Gamma, E'} : 2^N \rightarrow [0, \infty)$  that is defined by

$$f_d^{\Gamma, E'}(U) = \sum_{j: e_j \in E'} t(e_j) - \sum_{e \in g^{-1}(U) \cap E'} t(e). \quad (5)$$

Note that in case  $d$  is a minimal delivery tour of  $U$  in  $\Gamma$  we have for the corresponding delivery game  $(N, c)$  that  $c(U) = f_d^{\Gamma, E}(U)$  (cf. (4)). For a delivery problem that arises from a circuit the following lemma shows that in case  $e^* \notin d_1$ , then there are only three possible tours for  $d_2$ . Note that in the following lemma we have three possibilities in which way  $e^*$  can be surrounded



by a coalition  $S$ . The most important one is case (a) in lemma 1). The other two cases, (case (b) and (c) in lemma 1), will not be considered in this paper, but will be given in lemma 1 for completeness. (See also the remark after the proof of lemma 1).

**Lemma 1** Let  $\Gamma = (N, (V, E, i, v_0), g, t) \in DE(N)$  be a delivery problem and let  $(V, E, i)$  be the circuit  $v_0, e^1, v_1, \dots, v_{m-1}, e^m, v_0$ . Let  $k$  be such that  $e^k = e^*$ . (a) If there exists  $s_1, s_2 \in S, s_1 < k < s_2$  such that  $d_1 := v_0, e^1, \dots, e^{s_1}, v_{s_1}, e^{s_1}, \dots, e^1, v_0, e^m, \dots, e^{s_2}, v_{s_2-1}, e^{s_2}, \dots, e^m, v_0$  is the unique minimal delivery tour of  $S$ , then  $d_2$  is one of the following three tours:

$$v_0, e^1, \dots, e^k, v_k, e^k, \dots, e^1, v_0, e^m, \dots, e^{s_2}, v_{s_2-1}, e^{s_2}, \dots, e^m, v_0 \quad (6)$$

$$v_0, e^1, \dots, e^{s_1}, v_{s_1}, e^{s_1}, \dots, e^1, v_0, e^m, \dots, e^k, v_{k-1}, e^k, \dots, e^m, v_0 \quad (7)$$

$$v_0, e^1, \dots, e^m, v_0 \quad (8)$$

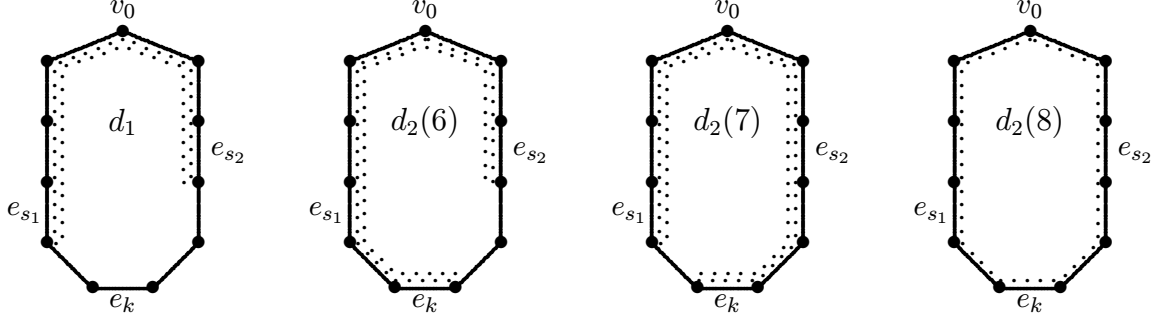


figure 3

(b) If there exists  $s_1 \in S, s_1 < k$  such that  $d_1 := v_0, e^1, \dots, e^{s_1}, v_{s_1}, e^{s_1}, \dots, e^1, v_0$  is the unique minimal  $S$ -tour, then  $d_2$  is one of the following three tours:

$$v_0, e^1, \dots, e^k, v_k, e^k, \dots, e^1, v_0 \quad (9)$$

$$v_0, e^1, \dots, e^{s_1}, v_{s_1-1}, e^{s_1}, \dots, e^1, v_0, e^m, \dots, e^k, v_{k-1}, e^k, \dots, e^m, v_0 \quad (10)$$

$$v_0, e^1, \dots, e^m, v_0 \quad (11)$$

(c) If there exists  $s_2 \in S, s_2 > k$  such that  $d_1 := v_0, e^m, \dots, e^{s_2}, v_{s_2}, e^{s_2}, \dots, e^m, v_0$  is the unique minimal  $S$ -tour, then  $d_2$  is one of the following three tours:

$$v_0, e^m, \dots, e^k, v_k, e^k, \dots, e^m, v_0 \quad (12)$$

$$v_0, e^1, \dots, e^k, v_k, e^k, \dots, e^1, v_0, e^m, \dots, e^{s_2}, v_{s_2}, e^{s_2}, \dots, e^m, v_0 \quad (13)$$

$$v_0, e^1, \dots, e^m, v_0 \quad (14)$$

PROOF: Since the proofs of the cases (b) and (c) are similar to (a), we only provide the proof of (a). Let  $u_1, u_2 \in S$  such that  $u_1 < u_2 \leq s_1$  or  $s_2 \leq u_1 < u_2$ . Moreover, for each  $k \in N$  with  $u_1 < k < u_2$  we must have  $k \notin S$ . Consider the delivery tour  $d^* := v_0, e^1, \dots, e^{u_1}, v_{u_1}, e^{u_1}, \dots, e^1, v_0, e^m, \dots, e^{u_2}, v_{u_2-1}, e^{u_2}, \dots, e^m, v_0$ . We will show that for coalition  $S \cup \{j\}$  the costs of the delivery tour  $v_0, e^1, \dots, e^m, v_0$  are smaller than the costs of  $d^*$ . Hence, the only possible minimal  $S$ -tours are the tours (6), (7) or (8). We may assume that

$u_1 < u_2 \leq s_1$ . First we derive an inequality which follows from the fact that  $d^*$  is a delivery tour of  $S$  but not a minimal one. We have

$$\begin{aligned} 0 &< f_{d^*}^{\Gamma,E}(S) - f_{d_1}^{\Gamma,E}(S) \\ &= \sum_{p=1}^{u_1} 2t(e^p) + \sum_{p=u_2}^m 2t(e^p) - \sum_{p=1}^{s_1} 2t(e^p) - \sum_{p=s_2}^m 2t(e^p) \\ &= \sum_{p=s_1+1}^{s_2-1} 2t(e^p) - \sum_{p=u_1+1}^{u_2-1} 2t(e^p) \end{aligned}$$

This leads to the following inequality

$$\sum_{p=u_1+1}^{u_2-1} 2t(e^p) < \sum_{p=s_1+1}^{s_2-1} 2t(e^p) \quad (15)$$

Take now for  $d_2$  expression (8). Then

$$\begin{aligned} f_{d_2}^{\Gamma,E}(S \cup \{j\}) &= \sum_{p=1}^m t(e^p) - \sum_{e \in g^{-1}(S \cup \{j\})} t(e) \\ &= \sum_{p=1}^{u_1} t(e^p) + \sum_{p=u_1+1}^{u_2-1} t(e^p) + \sum_{p=u_2}^m t(e^p) - \sum_{e \in g^{-1}(S \cup \{j\})} t(e) \\ &< \sum_{p=1}^{u_1} t(e^p) + \sum_{p=s_1+1}^{s_2-1} t(e^p) + \sum_{p=u_2}^m t(e^p) - \sum_{e \in g^{-1}(S \cup \{j\})} t(e) \\ &\leq \sum_{p=1}^{u_1} 2t(e^p) + \sum_{p=u_2}^m 2t(e^p) - \sum_{e \in g^{-1}(S \cup \{j\})} t(e) = f_{d^*}^{\Gamma,E}(S \cup \{j\}) \end{aligned}$$

where the first inequality follows from (15). Hence,  $d^*$  is not a minimal delivery tour of  $S \cup \{j\}$ . This gives that the only three possible minimal delivery tours of coalition  $S \cup \{j\}$  are given by (6), (7) or (8). For the special case that  $u_1 = s_1$  or  $u_2 = s_2$  we have to pick for  $d^* := v_0, e^m, \dots, e^{u_1}, v_{u_1}, e^{u_1}, \dots, e^m, v_0$  or  $d^* := v_0, e^1, \dots, e^{u_2}, v_{u_2-1}, e^{u_2}, \dots, e^m, v_0$ , respectively. Now we can show in a similar way that  $d^*$  is not optimal.  $\square$  **Remark:**

In the following we restrict attention to case (a) of lemma 1, i.e. we assume that the edge  $e^*$  is surrounded by two edges of  $S$ . This is possible since the proofs that follow can be elaborated to the cases (b) and (c) of lemma 1 using the same techniques that will be demonstrated for case (a).

The following theorem shows that a delivery game corresponding to a delivery problem that arises from a circuit graph is concave.

**Theorem 2** Let  $\Gamma = (N, (V, E, i, v_0), g, t) \in DE(N)$  be a delivery problem and let  $(V, E, i)$  be the circuit  $v_0, e^1, v_1, \dots, v_{m-1}, e^m, v_0$ . Then the corresponding delivery game  $(N, c)$  is concave.

PROOF: Let  $k$  be such that  $e^k = e^*$ . Then we have to show that

$$c(T \cup \{j\}) - c(T) \leq c(S \cup \{j\}) - c(S). \quad (16)$$

for  $S \subset T \subset N \setminus \{j\}$ . We have to consider four cases: (i)  $e^k \in d_1, e^k \in d_3$ . It follows that  $d_1 = d_2$  and  $d_3 = d_4$ . Consequently,

$$c(S \cup \{j\}) - c(S) = c(T \cup \{j\}) - c(T) = -t(e^k)$$

Hence, relation (16) is satisfied. (ii)  $e^k \notin d_1, e^k \in d_3$ . Since  $e^k \notin d_1$ , we have that the delivery tour  $d_1$  is equal to  $v_0, e^1, \dots, e^{s_1}, v_{s_1}, e^{s_1}, \dots, e^1, v_0, e^m, \dots, e^{s_2}, v_{s_2-1}, e^{s_2}, \dots, e^m, v_0$  where  $s_1 < k < s_2$ . Since,  $e^k \in d_2$  we have by lemma 1 (a) that  $d_2$  is one of the tours given by (6), (7) or (8). Choosing expression (6) yields

$$\begin{aligned} c(S \cup \{j\}) - c(S) &= \left\{ \sum_{p=1}^k 2t(e^p) + \sum_{p=s_2}^m 2t(e^p) - \sum_{e \in g^{-1}(S \cup \{j\})} t(e) \right\} \\ &\quad - \left\{ \sum_{p=1}^{s_1} 2t(e^p) + \sum_{p=s_2}^m 2t(e^p) - \sum_{e \in g^{-1}(S)} t(e) \right\} \\ &= -t(e^k) + \sum_{p=s_1+1}^{k-1} 2t(e^p) \geq -t(e^k) = c(T \cup \{j\}) - c(T) \end{aligned}$$

where the last equality follows from  $d_3 = d_4$ . Hence, (16) is satisfied. An analogous result is obtained when we choose (7) for  $d_2$ . Finally, choose (8) for  $d_2$ . Then

$$\begin{aligned} c(S \cup \{j\}) - c(S) &= \left\{ \sum_{p=1}^m t(e^p) - \sum_{e \in g^{-1}(S \cup \{j\})} t(e) \right\} \\ &\quad - \left\{ \sum_{p=1}^{s_1} 2t(e^p) + \sum_{p=s_2}^m 2t(e^p) - \sum_{e \in g^{-1}(S)} t(e) \right\} \\ &\geq \left\{ \sum_{p=1}^m t(e^p) - \sum_{e \in g^{-1}(S \cup \{j\})} t(e) \right\} - \left\{ \sum_{p=1}^m t(e^p) - \sum_{e \in g^{-1}(S)} t(e) \right\} = -t(e^k) \end{aligned}$$

The inequality follows from the fact that  $e^1, \dots, e^m$  is a delivery tour of  $S$ , but not necessarily a minimal one. Hence, (16) is satisfied. (iii)  $e^k \notin d_1, e^k \notin d_3$ . Take for  $d_1$  the same tour as in (ii). Consequently,  $d_2$  is equal to one of the expressions (6), (7) or (8). Moreover, since  $e^k \notin d_3$  we have that  $d_3$  is equal to  $v_0, e^1, \dots, e^{t_1}, v_{t_1}, e^{t_1}, \dots, e^1, e^m, \dots, e^{t_2}, v_{t_2-1}, e^{t_2}, \dots, e^m, v_0$  where  $s_1 \leq t_1 < k < t_2 \leq s_2$ . Since,  $e^k \in d_4$  we have that  $d_4$  is one of the following tours:

$$v_0, e^1, \dots, e^k, v_k, e^k, \dots, e^1, v_0, e^m, \dots, e^{t_2}, v_{t_2-1}, e^{t_2}, \dots, e^m, v_0 \quad (17)$$

$$v_0, e^1, \dots, e^{t_1}, v_{t_1}, e^{t_1}, \dots, e^1, v_0, e^m, \dots, e^k, v_{k-1}, e^k, \dots, e_m, v_0 \quad (18)$$

$$v_0, e^1, \dots, e^m, v_0 \quad (19)$$

Let  $d_2$  be equal to (6). Then

$$\begin{aligned} c(S \cup \{j\}) - c(S) &= -t(e^k) + \sum_{p=s_1+1}^{k-1} 2t(e^p) \\ &\geq -t(e^k) + \sum_{p=t_1+1}^{k-1} 2t(e^p) \geq c(T \cup \{j\}) - c(T). \end{aligned}$$

The first inequality holds by  $s_1 \leq t_1$  and the second inequality holds since (17) is not necessarily a minimal tour of  $T \cup \{j\}$ . In case  $d_2$  is equal to (7) we obtain a similar result. Finally, take  $d_2$  equal to (8), then

$$\begin{aligned} c(S \cup \{j\}) - c(S) &= \left\{ \sum_{p=1}^m t(e^p) - \sum_{e \in g^{-1}(S \cup \{j\})} t(e) \right\} \\ &\quad - \left\{ \sum_{p=1}^{s_1} 2t(e^p) + \sum_{p=s_2}^m 2t(e^p) - \sum_{e \in g^{-1}(S)} t(e) \right\} \\ &\geq \left\{ \sum_{p=1}^m t(e^p) - \sum_{e \in g^{-1}(T \cup \{j\})} t(e) \right\} \\ &\quad - \left\{ \sum_{p=1}^{t_1} 2t(e^p) + \sum_{p=t_2}^m 2t(e^p) - \sum_{e \in g^{-1}(T)} t(e) \right\} \\ &\geq c(T \cup \{j\}) - c(T) \end{aligned}$$

where the last inequality follows from the fact that  $e^1, \dots, e^m$  is a delivery tour of  $T \cup \{j\}$ , but not necessarily a minimal delivery tour. (iv)  $e^k \in d_1, e^k \notin d_3$  We will prove that this case is not possible by showing that if  $e^k \notin d_3$  then  $e^k \notin d_1$ . Take for  $d_3$  the same tour as in (iii) and note that we may assume that it is the unique minimal delivery tour. Suppose  $\hat{d} = v_0, e^1, \dots, e^{u_1}, v_{u_1}, e^{u_1}, \dots, e^1, v_0, e^m, \dots, e^{u_2}, v_{u_2-1}, e^{u_2}, \dots, e^m, v_0$  is a minimal delivery tour of  $S$  that contains  $e^k$ . Then  $u_2 \geq u_1 \geq t_2$  or  $t_1 \geq u_2 \geq u_1$  since  $\{e^{t_1+1}, \dots, e^{t_2-1}\} \cap T = \emptyset$ . Since  $d_3$  is also a delivery tour of  $S$  we have by the minimality of  $\hat{d}$  that

$$\begin{aligned} 0 &> f_{\hat{d}}^{\Gamma, E}(S) - f_{d_3}^{\Gamma, E}(S) \\ &= \sum_{p=1}^{u_1} 2t(e^p) + \sum_{p=u_2}^m 2t(e^p) - \sum_{p=1}^{t_1} 2t(e^p) - \sum_{p=t_2}^m 2t(e^p) \\ &= \sum_{p=t_1+1}^{t_2-1} 2t(e^p) - \sum_{p=u_1+1}^{u_2-1} 2t(e^p) \end{aligned}$$

Hence, from the above calculations we have the following inequality:

$$\sum_{p=u_1+1}^{u_2-1} t(e^p) > \sum_{p=t_1+1}^{t_2-1} t(e^p) \quad (20)$$

From the facts that  $v_0, e^1, \dots, e^m, v_0$  is a delivery tour of  $T$  and  $d_3$  is the unique minimal delivery tour of  $T$  we have

$$\begin{aligned} 0 &> f_{d_3}^{\Gamma, E}(T) - f_{v_0, e^1, \dots, e^m, v_0}^{\Gamma, E}(T) = \sum_{p=1}^{t_1} 2t(e^p) + \sum_{p=t_2}^m 2t(e^p) - \sum_{p=1}^m t(e^p) \\ &= \sum_{p=1}^{t_1} t(e^p) + \sum_{p=t_2}^m t(e^p) - \sum_{p=t_1+1}^{t_2-1} t(e^p). \end{aligned}$$

Hence, we obtain the following inequality:

$$\sum_{p=t_1+1}^{t_2-1} t(e^p) > \sum_{p=1}^{t_1} t(e^p) + \sum_{p=t_2}^m t(e^p) \quad (21)$$

In case  $u_1 \geq t_2$  we have  $\sum_{p=t_2}^m t(e^p) \geq \sum_{p=u_1+1}^{u_2-1} t(e^p)$ . Combining this result with inequality (21) we have that

$$\sum_{p=t_1+1}^{t_2-1} t(e^p) > \sum_{p=t_2}^m t(e^p) \geq \sum_{p=u_1+1}^{u_2-1} t(e^p)$$

which contradicts inequality (20). Similarly we get a contradiction in case  $u_2 \leq t_1$ . Hence, we may conclude that we have a contradiction with the assumption that  $\hat{d}$  is a minimal delivery tour for  $S$ . Consequently,  $v_0, e^1, \dots, e^m, v_0$  is the only possible minimal delivery tour of  $S$  that contains  $e^*$ . Then

$$0 < f_{d_3}^{\Gamma, E}(S) - f_{v_0, e^1, \dots, e^m, v_0}^{\Gamma, E}(S) = f_{d_3}^{\Gamma, E}(T) - f_{v_0, e^1, \dots, e^m, v_0}^{\Gamma, E}(T) < 0$$

and again we have a contradiction. So,  $e^k$  is not contained in any minimal delivery tour of  $S$  whenever  $e^k \notin d_3$ .  $\square$

The following theorem shows that a delivery game corresponding to a delivery problem that arises from a graph existing of one circuit and one bridge that is not connected to the post office is concave.

**Theorem 3** *Let  $\Gamma = (N, (V, E, i, v_0), g, t)$  be a delivery problem and let  $(V, E, i)$  be a connected graph consisting of the circuit  $v_0, e^1, \dots, e^m, v_0$  and the bridge  $e^{m+1}$  that is not connected to  $v_0$ . Then the corresponding delivery game  $(N, c)$  is concave.*

PROOF: Suppose that  $e^{m+1}$  is connected to  $v_p$  with  $p \neq 0$ . Consider the delivery problem  $\Gamma' = (N, (V', E, i', v_0), g, t')$  where  $(V', E, i')$  is the circuit  $v_0, e^1, \dots, e^p, v_p, e^{m+1}, v^*, e^{p+1}, v_{p+1}, \dots, e^m, v_0$ . Further,  $t'(e^p) = t(e^p)$  for all  $p \in \{1, \dots, m\}$

and  $t'(e^{m+1}) = 0$ . Let  $d'_3$  be a minimal delivery tour of  $T$  in  $\Gamma'$  and let  $d'_4$  be a minimal delivery tour of  $T \cup \{j\}$  in  $\Gamma'$ . Then we have that

$$f_{d_3}^{\Gamma, E}(T) = \begin{cases} f_{d'_3}^{\Gamma', E}(T) & \text{if } g(e^{m+1}) \notin T \\ f_{d'_3}^{\Gamma', E}(T) + t(e^{m+1}) & \text{if } g(e^{m+1}) \in T \end{cases}$$

and

$$f_{d_4}^{\Gamma, E}(T \cup \{j\}) = \begin{cases} f_{d'_4}^{\Gamma', E}(T \cup \{j\}) & \text{if } g(e^{m+1}) \notin T \cup \{j\} \\ f_{d'_4}^{\Gamma', E}(T \cup \{j\}) + t(e^{m+1}) & \text{if } g(e^{m+1}) \in T \cup \{j\} \end{cases}$$

Let  $(N, c')$  be the delivery game corresponding to  $\Gamma'$ , then

$$c'(T \cup \{j\}) - c'(T) = c(T \cup \{j\}) - c(T) \quad (22)$$

Using the same arguments for coalition  $S$  we have that

$$c'(S \cup \{j\}) - c'(S) = c(S \cup \{j\}) - c(S) \quad (23)$$

Since the graph in  $\Gamma'$  is a circuit theorem 2 yields that  $(N, c')$  is a concave game. The concavity of  $(N, c)$  then follows from the expressions (22) and (23).  $\square$

A graph  $(V, E, i)$  is called a bridge-connected circuit graph if all the components of the graph  $(V, E - B(G), i|_{E-B(G)})$  are circuits or single vertices. The class of delivery problems that arises from bridge-connected circuit graphs is denoted by  $DC(N)$ . The following theorem will show that a delivery game corresponding to a delivery problem  $\Gamma \in DC(N)$  is concave. Before we can proof this theorem we need three lemmata which give some relations between delivery tours of different coalitions.

The first lemma shows that minimal delivery tours of  $T$  and  $T \cup \{j\}$  coincide on the set of followers of a bridge  $b \in B(G)$  if  $e^*$  is not a follower of that bridge. Here,  $e \in E$  is a follower of a bridge  $b$  with respect to  $v_0$  if and only if each path  $v_0, e_1, \dots, e_k, v_k$  that contains  $e$  also contains  $b$ . The set of followers of  $b$  will be denoted by  $F_b(G, v_0)$ .

**Lemma 2** *Let  $\Gamma = (N, (G, v_0), g, t) \in DC(N)$ . Let  $b \in B(G)$  be such that  $e^* \notin F_b(G, v_0)$ , then  $f_{d_3}^{\Gamma, F_b(G, v_0)}(T) = f_{d_4}^{\Gamma, F_b(G, v_0)}(T \cup \{j\})$ .*

PROOF: If  $F_b(G, v_0) \cap T = \emptyset$  then  $d_3$  and  $d_4$  will not visit  $F_b(G, v_0)$ . Hence, we may assume that  $F_b(G, v_0) \cap T \neq \emptyset$ . Since  $e^* \notin F_b(G, v_0)$  we have that  $d_3$  and  $d_4$  have to visit the same edges of  $T$  in  $F_b(G, v_0)$ . Consequently, both tours will coincide on  $F_b(G, v_0)$ .  $\square$

The next lemma shows that minimal delivery tours of  $T$  and  $T \cup \{j\}$  coincide on the predecessors of a bridge  $b$  if coalition  $T$  has a non-empty intersection with  $F_b(G, v_0)$  and  $e^*$  is a follower of  $b$ .

**Lemma 3** Let  $\Gamma = (N, (G, v_0), g, t) \in DC(N)$ . Then for each  $b \in B(G)$  such that  $T \cap F_b(G, v_0) \neq \emptyset$  and  $e^* \in F_b(G, v_0)$  we have that

$$f_{d_3}^{\Gamma, E - (F_b(G, v_0) - \{b\})}(T) = f_{d_4}^{\Gamma, E - (F_b(G, v_0) - \{b\})}(T \cup \{j\})$$

PROOF: Both delivery tours  $d_3$  and  $d_4$  have to visit the same edges in  $E - F_b(G, v_0)$  since  $e^* \in F_b(G, v_0)$ . Moreover, both tours have to visit bridge  $b$  since  $F_b(G, v_0) \cap T \neq \emptyset$ . This implies that both tours have the same costs in  $E - (F_b(G, v_0) - \{b\})$ .  $\square$

The last lemma shows that the delivery tours of  $T \cup \{j\}$  and  $S \cup \{j\}$  coincide on the followers of  $b$ , excluded that bridge, if no player of  $T$ , and consequently no player of  $S$ , is a follower of  $b$ .

**Lemma 4** Let  $\Gamma = (N, (G, v_0), g, t) \in DC(N)$  and let  $b \in B(G)$ . If  $T \cap (F_b(G, v_0) - \{b\}) = \emptyset$  then  $f_{d_2}^{\Gamma, F_b(G, v_0) - \{b\}}(S \cup \{j\}) = f_{d_4}^{\Gamma, F_b(G, v_0) - \{b\}}(T \cup \{j\})$ .

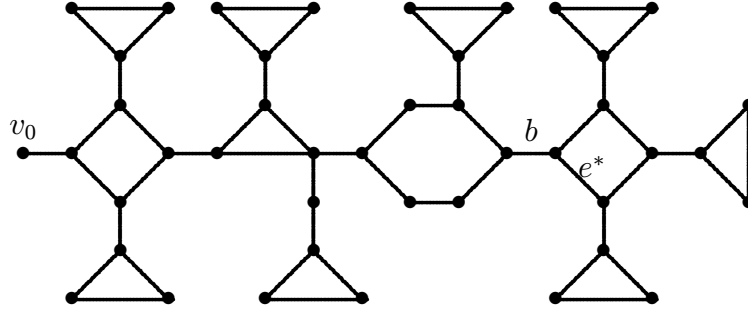
PROOF: If  $e^* \notin F_b(G, v_0)$  both delivery tours will not visit  $F_b(G, v_0)$ . If  $e^* \in F_b(G, v_0)$  both delivery tours will choose from  $b$  a shortest path to  $e^*$  since no other edges have to be delivered in  $F_b(G, v_0)$ . Hence, in both cases the delivery tours  $d_2$  and  $d_4$  have the same costs on  $F_b(G, v_0)$ .  $\square$

Before we can give the proof of the following theorem we need the notion of a bridge-connected line graph. This is a bridge-connected circuit graph such that each circuit is connected to at most two bridges.

**Theorem 4** Let  $\Gamma = (N, (G, v_0), g, t) \in DC(N)$ . Then the corresponding delivery game  $(N, c)$  is concave.

PROOF: Lemma 2 implies that we can reduce the concavity problem to a delivery model corresponding to a bridge-connected line graph such that  $e^* \in F_b(G, v_0)$  and  $b$  is the extreme bridge of the bridge-connected line graph. This reduction is illustrated in figure





4.

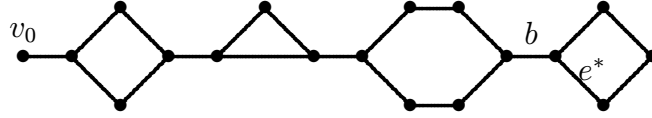
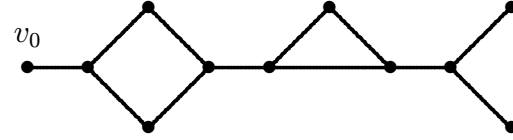


figure 4



Now we have to consider two cases: (i)  $T \cap F_b(G, v_0) \neq \emptyset$ .

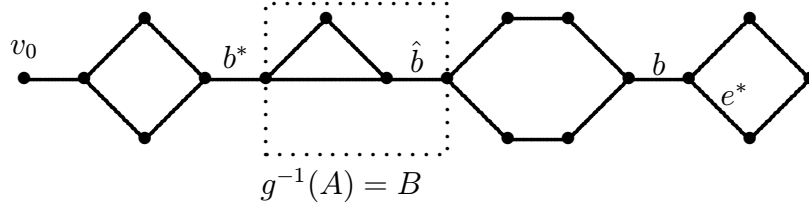
figure 5

If  $e^* = b$  then  $c(T \cup \{j\}) - c(T) = -t(e^*) \leq c(S \cup \{j\}) - c(S)$ . If, on the other hand,  $e^* \in F_b(G, v_0) - \{b\}$ , the following holds. Let  $F_b(G, v_0) - \{b\} = \{e^1, \dots, e^m\}$  and take  $v^*$  such that  $i(e^1) \cap i(e^m) \cap i(b) = v^*$ . Next, consider the delivery problem  $\bar{\Gamma} = (A, (V', F_b(G, v_0), i', v^*), g|_{F_b(G, v_0) - \{b\}}, t|_{F_b(G, v_0) - \{b\}})$  where  $A = g(F_b(G, v_0) - \{b\})$  and  $(V', F_b(G, v_0), i')$  is the graph that denotes the circuit described by  $F_b(G, v_0)$ . Let  $(N, \bar{c})$  be the delivery game corresponding to  $\bar{\Gamma}$ . Then

$$\begin{aligned} c(T \cup \{j\}) - c(T) &= f_{d_4}^{\Gamma, F_b(G, v_0) - \{b\}}(T \cup \{j\}) - f_{d_3}^{\Gamma, F_b(G, v_0) - \{b\}}(T) \\ &= \bar{c}((T \cap A) \cup \{j\}) - \bar{c}(T \cap A) \leq \bar{c}((S \cap A) \cup \{j\}) - \bar{c}(S \cap A) \end{aligned}$$

$$= f_{d_2}^{\Gamma, F_b(G, v_0) - \{b\}}(S \cup \{j\}) - f_{d_1}^{\Gamma, F_b(G, v_0) - \{b\}}(T) \leq c(S \cup \{j\}) - c(S)$$

Here, the first equality follows from lemma 3. The first inequality by the concavity of  $(N, \bar{c})$  (cf. theorem 2). (ii)  $T \cap F_b(G, v_0) = \emptyset$ . Let  $b^* \in B(G)$  be such that  $F_{b^*}(G, v_0) \cap T \neq \emptyset$  and  $T \cap F_{b'}(G, v_0) = \emptyset$  for all  $b' \in B(G) \cap (F_{b^*}(G, v_0) - \{b^*\})$ . Let  $\hat{b} \in B(G) \cap F_{b^*}(G, v_0)$  be the bridge that is connected to the set  $F_{b^*}(G, v_0) - \cup_{b' \in (B(G) - \{b^*\}) \cap F_{b^*}(G, v_0)} F_{b'}(G, v_0)$ . Hence,  $\hat{b}$  is the closest bridge that follows  $b^*$  (see figure



6).

figure 6

Let  $B := (F_{b^*}(G, v_0) \cup \hat{b}) - (\{b^*\} \cup F_{\hat{b}}(G, v_0)) = \{\hat{b}, e^1, \dots, e^m\}$  and take  $v^*$  such that  $i(e^1) \cap i(e^m) \cap i(b) = v^*$ . Let  $(V', B, i')$  be the graph that arises from the set consisting of the edges of  $B$ . Consider the delivery problem  $\bar{\Gamma} = (A, (V', B, i', v^*), g|_B, t|_B)$  where  $A = g(B)$ . Let  $(N, c')$  be the delivery game corresponding to  $\bar{\Gamma}$  and let  $g(\hat{b}) = j^*$ . Then

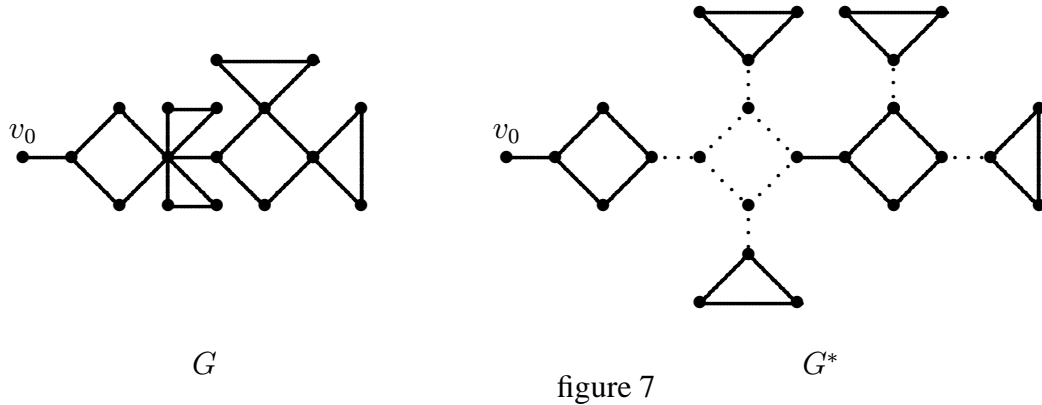
$$\begin{aligned} & c(T \cup \{j\}) - c(T) \\ &= c'((T \cap A) \cup \{j^*\}) - c'(T \cap A) + f_{d_4}^{\Gamma, F_{\hat{b}}(G, v_0) - \hat{b}}(T \cup \{j\}) \\ &\leq c'((S \cap A) \cup \{j^*\}) - c'(S \cap A) + f_{d_4}^{\Gamma, F_{\hat{b}}(G, v_0) - \hat{b}}(T \cup \{j\}) \\ &= c'((S \cap A) \cup \{j^*\}) - c'(S \cap A) + f_{d_4}^{\Gamma, F_{\hat{b}}(G, v_0) - \hat{b}}(S \cup \{j\}) \\ &= c(S \cup \{j\}) - c(S) \end{aligned}$$

where the first inequality follows from the concavity of  $(N, c')$  (cf. Theorem 3). The second equality follows from lemma 4.  $\square$

Finally we will show that delivery games corresponding to delivery problems that arise from bridge-connected cyclic graphs are concave. Recall that bridge-connected cyclic graphs are

the graphs that after removing the bridges only have cycles or single vertices as components. A cycle is a circuit or the union of edge-disjoint circuits. It is shown that each delivery game which corresponds to a delivery model arising from a bridge-connected cyclic graph is contained in a delivery game arising from a bridge-connected circuit graph. Therefore, we consider a procedure that extends a bridge-connected cyclic graph to a bridge-connected circuit graph. Consider all kissing points in the bridge-connected cyclic graph. These are the vertices of a graph that are in the intersection of at least two circuits and may be connected to some bridges. This implies that in case a kissing point is removed in a graph we obtain a disconnected graph. The first step in this procedure is to consider all kissing points that are the intersection of two circuits. These circuits are split by replacing such a kissing point by a bridge. In step two we consider the kissing points that are the intersection of at least three circuits. Then we replace such a kissing point by a circuit in which the number of vertices is equal to the number of circuits and bridges incident to that kissingpoint. Then we have new kissing points, but all these kissingpoints are incident with at most two circuits. We repeat now the first step of the procedure. This procedure, which results in a bridge-connected circuit graph, is illustrated in the following example.

**Example 3** The graph  $G$  is a bridge-connected cyclic graph and the graph  $G^*$  is the bridge-connected circuit graph that arises from  $G$  by the above described procedure.



Let  $(V, E, i)$  be a bridge-connected cyclic graph and let  $(\bar{V}, \bar{E}, \bar{i})$  be the extension. Then  $\bar{\Gamma} = (M, (\bar{V}, \bar{E}, \bar{i}, v_0), \bar{g}, \bar{t})$  is called the minimal extension of the delivery problem  $\Gamma = (N, (V, E, i, v_0), g, t)$  if  $\bar{g}(e) = g(e)$  for all  $e \in E$ ,  $\bar{t}(e) = t(e)$  for all  $e \in E$  and  $\bar{t}(e) = 0$  for all  $e \in \bar{E} - E$ . Note that  $N \subset M$ .

**Lemma 5** Let  $\Gamma = (N, (V, E, i, v_0), g, t)$  be a delivery problem and let  $\bar{\Gamma} =$

$(M, (\overline{V}, \overline{E}, \bar{i}, v_0), \overline{g}, \bar{t})$  be the minimal extension of  $\Gamma$ . Let  $(N, c)$  be the delivery game corresponding to  $\Gamma$  and let  $(N, \bar{c})$  be the delivery game corresponding to  $\overline{\Gamma}$  then

$$c(S) = \bar{c}(S) \quad \text{for all } S \subset N.$$

PROOF: If in  $\Gamma$  a kissing point is visited in a minimal delivery tour of a coalition  $S$  to visit another circuit then the corresponding minimal delivery tour of  $S$  in  $\overline{\Gamma}$  has to visit the new bridges and some parts of a new cycle between these two circuits. Since these new edges have travel costs equal to zero we have that the costs of a minimal delivery tour of  $S$  in  $\Gamma$  coincides with the costs of the corresponding delivery tour of  $S$  in  $\overline{\Gamma}$ .  $\square$  PROOF THEOREM 1: Follows immediately from theorem 4 and lemma 5.  $\square$

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